

Physics of crypto-Hermitian and/or cryptosupersymmetric field theories

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Abstract

We discuss non-Hermitian field theories where the spectrum of the Hamiltonian involves only real energies. We make three observations. *(i)* The theories obtained from supersymmetric theories by non-anticommutative deformations belong in many cases to this class. *(ii)* When the deformation parameter is small, the deformed theory enjoys *the same* supersymmetry algebra as the undeformed one. Half of the supersymmetries are manifest and the existence of another half can be deduced from the structure of the spectrum. *(iii)* Generically, the conventionally defined S -matrix is not unitary for such theories.

1 Introduction

There exists a rich class of quantum systems whose Hamiltonian is apparently not Hermitian, but which involve only real energies in the spectrum. Such systems have been intermittently discussed in the literature since mid-seventies [1]. They attracted great interest after the beautiful paper [2], where it was shown that the Hamiltonians with a certain type of complex potentials, like

$$H = p^2 + ix^3, \quad (1)$$

have real discrete spectrum.

It was observed in [3] that all such Hamiltonians are "crypto-Hermitian", i.e. can be represented in the form

$$H = e^R \tilde{H} e^{-R} \quad (2)$$

with Hermitian \tilde{H} . The matrix e^R is generically not unitary and hence H is not Hermitian, but the spectrum of H is the same as for \tilde{H} and involves only real eigenvalues. The similarity transformation (2) amounts to redefining the metric in Hilbert space. The Hamiltonian H is not Hermitian with respect to the standard metric, but it is Hermitian with respect to a redefined one.

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There is, however, a price that one has to pay for this. First, the transformation matrix R is typically not local. For the Hamiltonian (1), it involves inverse powers of momentum. For the Hamiltonian $H = p^2 + x^2 + igx^3$, the momentum does not show up in the denominator, but the matrix R represents an infinite series in g with growing powers of momentum. Second, the observables x, p , which are Hermitian with respect to the standard metric, are not Hermitian with respect to the new one and could hardly be interpreted as the coordinate and momentum of a physical particle.

For the systems like (1) with discrete real spectrum, the latter circumstance might be not so relevant. The spectral problem for the Hamiltonian (1) *can* be solved in a certain complex region of x . The solution is nontrivial, interesting, and one can be satisfied with this not asking the question of what is the physical interpretation of x . The situation is different, however, for cryptoreal systems with continuous spectrum, for example, for the PT-symmetric square well potential

$$V(x) = \begin{cases} 0, & |x| > a \\ iC, & 0 < x \leq a \\ -iC, & -a < x \leq 0 \end{cases} \quad (3)$$

As was discussed in [4, 5], in spite of the fact that such Hamiltonian is cryptoreal (its spectrum is real and the metric with respect to which the Hamiltonian is Hermitian exists), the scattering matrix *is* not unitary if defined in a usual way as the transition amplitude between ingoing and outgoing plane waves. Cryptoreality dictates that a basis where the evolution operator is unitary exists, but the states of this basis do not have a simple physical interpretation. This difficulty is an evident manifestation of the fact that the Hamiltonian H and the observable x cannot be made Hermitian simultaneously.

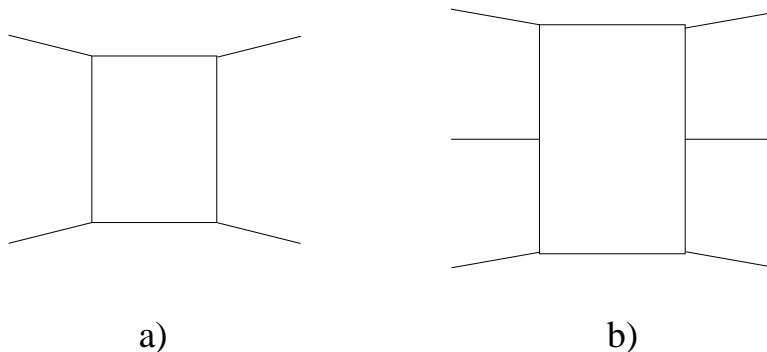


Figure 1: Forward scattering amplitudes in $i\phi^3$ theory.

We would like to make an obvious remark that the same applies to PT-symmetric and other cryptoreal field theories. The simplest example of the latter is the theory of a real scalar field $\phi(x)$ with the Lagrangian [6]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 + i\gamma \phi^3 . \quad (4)$$

The corresponding Hamiltonian is Hermitian with respect to the metric involving the transformation matrix

$$R = \gamma \int d\mathbf{x} d\mathbf{y} d\mathbf{z} (M_{\mathbf{xyz}} \pi_{\mathbf{x}} \pi_{\mathbf{y}} \pi_{\mathbf{z}} + N_{\mathbf{xyz}} \pi_{\mathbf{x}} \phi_{\mathbf{y}} \phi_{\mathbf{z}}) + O(\gamma^3) \quad (5)$$

with complicated nonlocal kernels $M_{\mathbf{xyz}}, N_{\mathbf{xyz}}$ ($\pi_{\mathbf{x}}$ are the canonical momenta). That means that, when the theory (4) is regulated in the infrared and is put in a finite spatial box, the spectrum of the corresponding Hamiltonian is discrete and real. On the other hand, the most relevant question we usually ask about a quantum field system is not what is its spectrum in finite box, but rather what is its S matrix - a set of transition amplitudes between standard *in* and *out* states, the eigenstates of the free Hamiltonian. For the theory (4), such S matrix is not unitary. Indeed, for a unitary theory, the imaginary part of all forward scattering amplitudes should be positive. For the $2 \rightarrow 2$ amplitude, it is still the case at least perturbatively — the analytic expression for the 1-loop amplitude depicted in Fig. 1a is the same as in the theory $\gamma\phi^3$ with real γ that is unitary at perturbative level. But the 1-loop $3 \rightarrow 3$ amplitude depicted in Fig. 1b has an opposite sign compared to the same amplitude in the theory $\gamma\phi^3$. This violates unitarity.

2 Non-anticommutative WZ model.

The main subject of the present paper are so called non-anticommutative (NAC) supersymmetric theories. They were introduced first in Ref. [7]. Seiberg took the standard Wess-Zumino model

$$\begin{aligned} \mathcal{L} &= \int d^4\theta \bar{\Phi}\Phi + \left[\int d^2\theta \left(\frac{m\Phi^2}{2} + \frac{g\Phi^3}{3} \right) + \text{c.c.} \right] \\ &\equiv |\partial_\mu\phi|^2 + i\bar{\psi}\hat{\partial}\psi - |W(\phi)|^2 + [W'(\phi)\psi^2 + H.c.] \end{aligned} \quad (6)$$

with $W(\phi) = m\phi + g\phi^2$ and deformed it by introducing the nontrivial anticommutator

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \quad (7)$$

$C^{\alpha\beta} = C^{\beta\alpha}$, in the assumption that all other (anti)commutators vanish,

$$\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = [\theta^\alpha, x_\mu^L] = [\bar{\theta}^{\dot{\alpha}}, x_\mu^L] = [x_\mu^L, x_\nu^L] = 0. \quad (8)$$

Note that this all was written in the *chiral* basis, $x_\mu^L = x_\mu^{\text{central}} + i\theta\sigma_\mu\bar{\theta}$.

The anticommutator (7) introduces a constant self-dual tensor, which explicitly breaks Lorentz invariance. However, the deformed Lagrangian expressed in terms of the component fields proves still to be Lorentz invariant. Indeed, it is easy to find that the kinetic term $\int d^4\theta \bar{\Phi}\Phi$ is undeformed and the only extra piece comes from

$$\Delta\mathcal{L} = \frac{g}{3} \int d^2\theta \Phi * \Phi * \Phi - \frac{g}{3} \int d^2\theta \Phi^3 = -\frac{g}{3} \det \|C\| F^3, \quad (9)$$

F being the auxiliary field. It depends only on the scalar $\det \|C\|$ and is obviously Lorentz invariant. Adding the usual terms $F(m\phi + g\phi^2) + \bar{F}(m\bar{\phi} + \bar{g}\bar{\phi}^2)$ coming from superpotential and $F\bar{F}$ from the kinetic term, and expressing F and \bar{F} via ϕ and $\bar{\phi}$, we see that the undeformed potential $|m\phi + g\phi^2|^2$ acquires an extra holomorphic contribution $\propto g(m\bar{\phi} + \bar{g}\bar{\phi}^2)^3$.

This extra contribution is *complex*. This makes the Lagrangian and Hamiltonian complex. A generic complex Hamiltonian has complex spectrum, the corresponding evolution operator is not unitary, and the theory has little physical sense. This is the reason why people mainly considered up to now NAC models in Euclidean space only (where these problems do not arise) and were not interested in the corresponding Minkowski dynamics. We will show, however, that, even though the Hamiltonian of NAC WZ model does not look Hermitian, it is in fact crypto-Hermitian under a special choice of the deformation parameter and possesses a real spectrum.

This is exactly what was observed in Ref. [8] for another NAC deformed model, the Aldrovandy-Schaposnik model [9] (basically, this is a deformation of Witten's SQM model). We conjectured in Ref. [8] that crypto-Hermiticity of the deformed Hamiltonian holds also in the NAC WZ model. We confirm this conjecture here. Another conjecture of Ref. [8] that, in the WZ case, the spectrum of the deformed model is not shifted compared to the undeformed case is correct only when $g = 0$. Then the undeformed model is free and so is the deformed one. But the interacting model *is* deformed in a nontrivial way and its spectrum is shifted.

Our second observation is that the deformed model still has *four* conserved supercharges $Q_\alpha, \bar{Q}^{\dot{\beta}}$. Half of the supercharges (the supercharges Q_α under the standard convention) are the same as in the undeformed model, while the supercharges $\bar{Q}^{\dot{\beta}}$ are modified and acquire a rather complicated form. The spectrum of the theory includes 2 bosonic vacuum states and degenerate quartets of excited states, like in the undeformed WZ theory. In other words, the common lore that NAC deformations break half of supersymmetries is not correct (at least, it is not correct in the cases analyzed). All supersymmetries stay intact. Half of them are manifest and another half are realized in a complicated indirect way.

Again, this is very much similar to what happens in the AS model where one of the supercharges has the same form as in undeformed Witten's model while another one is modified.

Generically, Q_α and $\bar{Q}^{\dot{\beta}}$ are not adjoint to each other and the Hamiltonian is not Hermitian. However, when the deformation parameter is chosen in a special way, the spectrum stays real and the Hamiltonian is thus crypto-Hermitian. On the other hand, the deformed field interacting theory is not unitary in the same sense and by the same reasons as $i\phi^3$ theory discussed above. For the free WZ model, the deformed model is physically equivalent to the undeformed one and its S -matrix is trivial.

To derive all the results mentioned above in a manifest way, let us consider the dimensionally reduced system and assume that the fields do not depend on spatial coordinates.

The reduced Hamiltonian is

$$H = \bar{\pi}\pi + \bar{\phi}\phi + g\phi^2\bar{\phi} + \bar{g}\bar{\phi}^2\phi + g\bar{g}\bar{\phi}^2\phi^2 - (1 + 2g\phi)\psi_1\psi_2 - (1 + 2\bar{g}\bar{\phi})\bar{\psi}_2\bar{\psi}_1 + \beta(\bar{\phi} + \bar{g}\bar{\phi}^2)^3 \quad (10)$$

with $\bar{\psi}_\alpha \equiv \partial/\partial\psi_\alpha$ and $\beta = g \det \|C\|/3$ being the deformation parameter. For simplicity, we have set $m = 1$.

The wave functions for this Hamiltonian have four components, being represented as

$$\Psi(\phi, \bar{\phi}, \psi_\alpha) = A(\bar{\phi}, \phi) + B_\alpha(\bar{\phi}, \phi)\psi_\alpha + C(\bar{\phi}, \phi)\psi_1\psi_2. \quad (11)$$

In the undeformed case, the Hamiltonian (10) admits conserved supercharges

$$\begin{aligned} Q_\alpha &= \pi\psi_\alpha + i\epsilon_{\alpha\gamma}\bar{\psi}_\gamma(\bar{\phi} + \bar{g}\bar{\phi}^2), \\ \bar{Q}_\beta &= \bar{\pi}\bar{\psi}_\beta - i\epsilon_{\beta\delta}\psi_\delta(\phi + g\phi^2) \end{aligned} \quad (12)$$

with $\epsilon_{12} = 1$. They satisfy the usual $\mathcal{N} = 2$ SQM algebra

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0, \quad \{Q_\alpha, \bar{Q}_\beta\} = H\delta_{\alpha\beta} \quad (13)$$

Consider first the free Hamiltonian

$$H_0 = \bar{\pi}\pi + \bar{\phi}\phi - (\psi_1\psi_2 + \bar{\psi}_2\bar{\psi}_1). \quad (14)$$

This is the supersymmetric 2-dimensional oscillator and the wave functions can be found explicitly being expressed via certain Laguerre polynomials. Let the functions $|ln\rangle$ be the eigenfunctions of the bosonic Hamiltonian $\bar{\pi}\pi + \bar{\phi}\phi$ (l is the eigenvalue of the charge operator $i(\phi\pi - \bar{\phi}\bar{\pi})$ that commutes with H_0 , and n is the principal quantum number). The explicit expressions for first few levels are

$$\begin{aligned} |00\rangle &= \sqrt{\frac{2}{\pi}}e^{-\bar{\phi}\phi}, \quad |10\rangle = \sqrt{\frac{2}{\pi}}\phi e^{-\bar{\phi}\phi}, \quad |-10\rangle = \sqrt{\frac{2}{\pi}}\bar{\phi}e^{-\bar{\phi}\phi}, \\ |01\rangle &= \sqrt{\frac{2}{\pi}}(1 - 2\phi\bar{\phi})e^{-\bar{\phi}\phi}, \quad |20\rangle = \frac{2}{\sqrt{\pi}}\phi^2 e^{-\bar{\phi}\phi}, \quad |-20\rangle = \frac{2}{\sqrt{\pi}}\bar{\phi}^2 e^{-\bar{\phi}\phi}, \dots \end{aligned} \quad (15)$$

The spectrum of the full Hamiltonian (14) involves the fermionic states $\Psi^{\text{ferm}} = |ln\rangle\psi_\alpha$ with the energies $E_{ln}^F = |l| + 2n + 1$ and the bosonic states

$$\Psi^{\text{bos}} = |ln\pm\rangle \equiv \frac{1}{\sqrt{2}}(1 \pm \psi_1\psi_2)|ln\rangle \quad (16)$$

with the energies $E_{nl+}^B = |l| + 2n$, $E_{nl-}^B = |l| + 2n + 2$. There is a single vacuum state $|00+\rangle$, while the excited states come in quartets: the quartet of states

$$|00\rangle\psi_\alpha, \quad |\pm 10+\rangle \quad (17)$$

has the energy 1, there are two quartets of energy 2:

$$\left\{ |-20+\rangle, |-10\rangle\psi_\alpha, \frac{[|00-\rangle + |01+\rangle]}{\sqrt{2}} \right\} \text{ and } \left\{ \frac{[|00-\rangle - |01+\rangle]}{\sqrt{2}}, |10\rangle\psi_\alpha, |20+\rangle \right\}, \quad (18)$$

three quartets of energy 3, etc. The members of a quartet are produced from each other by the action of the supercharges (12)

Let us assume g and β to be nonzero but small and treat them perturbatively. To determine the energy shifts is a not so difficult exercise in standard quantum mechanics perturbation theory. One has to evaluate the graphs in Fig.2, where the charge -3 of the perturbation $\beta\bar{\phi}^3$ (in the lowest nontrivial order, we can set $g = 0$ in the last term in Eq.(10)) is compensated by three insertions of the perturbation $g\bar{\phi}\phi^2 - 2g\phi\psi_1\psi_2$ of charge 1, while the perturbations $\propto \bar{g}$ and $\propto \bar{g}g$ in Eq.(10) are not relevant to this order.

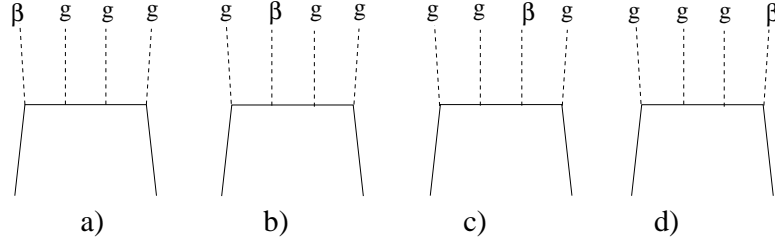


Figure 2: Graphs contributing to the energy shift $\propto \beta g^3$

A somewhat long, but straightforward calculation gives the result: the ground state energy is not shifted and stays zero *, while the first excited level is shifted by

$$\Delta E_1 = -\frac{155}{36}\beta g^3. \quad (19)$$

The shift is the same for the bosonic states $|\pm 10+\rangle$ and the fermionic states $|00\rangle\psi_\alpha$. Generically, the energy shift is complex, but, for real βg^3 , it is real.

The deformation $\propto W^3(\bar{\phi})$ follows from NAC machinery, but one can equally well consider simpler complex deformations $\Delta H = \beta_1\bar{\phi}$ and $\Delta H = \beta_2\bar{\phi}^2$. Again, the vacuum energy is not shifted, while the excited levels are shifted such that the quartets of the states are still degenerate. For the first excited quartet with energy $E_1 = 1$, one can calculate (it is the second order of perturbation theory for the deformation $\propto \beta_1$ and the third order for the deformation $\propto \beta_2$)

$$\Delta E_1(\beta_1) = -\beta_1 g; \quad \Delta E_1(\beta_2) = \frac{31}{18}\beta_2 g^2. \quad (20)$$

*The Witten index [10] of the interacting Wess-Zumino model is not 1 but 2, and the second vacuum state should appear. But the corresponding wave function lives at large values of $|\phi|$ and its presence cannot be detected perturbatively.

The zero vacuum energy and 4-fold degeneracy of the excited levels means that the deformed model still enjoys supersymmetry and the algebra (13) holds, though H is not necessarily Hermitian and Q_α and \bar{Q}_α are not necessarily conjugate to each other (cf. [11], [9]).

Speaking of the supercharge Q_α , it is still given by the expression in Eq.(12), which commutes with the deformed Hamiltonian. On the other hand, the commutator of the undeformed supercharge \bar{Q}_α with the deformed Hamiltonian does not vanish. In contrast to AS model where a simple expression for the deformed supercharge $\bar{Q}_\alpha^{\text{deformed}}$ can be written [9], we cannot do it in our case. By no means \bar{Q}_α can be obtained by complex conjugation of the supercharge Q_α . Indeed, a pair of complex conjugate supercharges would mean Hermiticity of Hamiltonian, but the Hamiltonian (10) is not Hermitian. The fact that its spectrum is real (when βg^3 is real) tells, however, that the Hamiltonian is crypto-Hermitian in the same sense as the AS Hamiltonian is. In particular, the operator R rotating the Hamiltonian to the manifestly Hermitian form should exist.

Even though explicit expressions for \bar{Q}_α are not known, one can argue that the quartet supersymmetric structure of the spectrum must hold without making explicit calculations. It can be reconstructed (at least, perturbatively [†]) using only Q_α and not \bar{Q}_α . Indeed, for each supersymmetric quartet of the eigenstates of the free Hamiltonian H_0 , a member Ψ annihilated by the action of \bar{Q}_α , but not Q_α , can be chosen. Three other members of the quartet are $Q_{1,2}\Psi$ and $Q_1Q_2\Psi$. Let $\tilde{\Psi}$ be the corresponding eigenstate of the full Hamiltonian (when β and g are small, one can be sure that such state exists). Then $\tilde{\Psi}$, $Q_\alpha\tilde{\Psi}$, and $Q^2\tilde{\Psi}$ represent a quartet of degenerate eigenstates of the interacting deformed Hamiltonian. Once the states are known, the matrix elements of \bar{Q}_α can be defined to be equal to the corresponding matrix elements in the free undeformed basis multiplied by $\sqrt{E_n^{\text{exact}}/E_n^{\text{free}}}$.

If you will, the theories of this kind (where supersymmetry is not manifest at the Lagrangian level, but is there as far as the structure of the spectrum is concerned) can be called *cryptosupersymmetric*. A very interesting question to be studied is whether and if so then how cryptosupersymmetry of deformed models (i.e. usual supersymmetry with complicated deformed supercharges) corresponds to the so called *twist-deformed supersymmetry* for conventional supersymmetry generators unravelled in Refs. [13]. (It was shown there that the conventional generators, which do not satisfy standard SUSY algebra in the deformed case, form a certain twisted Hopf quantum superalgebra.)

3 Discussion.

What conclusions concerning NAC field theories can be made on the basis of this analysis? If we put the theory in a finite spatial box and be interested in the spectrum of the Hamiltonian thus obtained, one can conjecture that its properties should be similar to the properties of the dimensionally reduced deformed WZ Hamiltonian:

[†]It would be very interesting to study the spectrum of the deformed Hamiltonian numerically. One cannot exclude a possibility that exceptional points [12] in the space of couplings appear such that the supersymmetric structure of the spectrum would be lost for large enough values of β, g .

- The ground state energy(ies) is(are) still zero (if supersymmetry is not spontaneously broken) and the 2^N degeneracy of the excited spectrum states should be kept.
- For certain values of the deformation parameters and the couplings, the spectrum of the deformed Hamiltonian should enjoy crypto-Hermiticity property.

However, a conventionally defined S -matrix is not unitary in deformed interacting NAC theories by the same token as it is not unitary in the theory $i\gamma\phi^3$. The complexity of Minkowski space Lagrangian strikes back at this point. This means that these theories cannot be attributed a *conventional* physical meaning. More studies of this question are necessary. Maybe even if S -matrix of the theory is not unitary, unitarity of its finite time finite box evolution operator (that follows from crypto-Hermiticity of the Hamiltonian) suffices to make the theory meaningful ? A positive answer to this question would mean a breakthrough in understanding of not only NAC theories, but also theories with higher derivatives in the Lagrangian. In Ref. [14], we argued that the fundamental Theory of Everything may be a theory of this kind. We address the reader to this paper and also to the papers [15] for discussions and speculations on this subject.

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